Cramér asymptotics for finite time first passage probabilities of general Lévy processes

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Abstract

We derive the exact asymptotics of $P(\sup_{u\leq t} X(u) > x)$ if x and t tend to infinity with x/t constant, for a general Lévy process X that admits exponential moments. The proof is based on a renewal argument and a two-dimensional renewal theorem of Höglund [9].

1 Introduction

The study of boundary crossing probabilities of Lévy processes has applications in many fields, including ruin theory (see e.g. Rolski et al. [13] and Asmussen [2]), queueing theory (see e.g. Borovkov [6] and Prabhu [11]), statistics (see e.g. Siegmund [15]) and mathematical finance (see e.g. Roberts and Shortland [12]).

As in many cases closed form expressions for (finite time) first passage probabilities are either not available or intractable, a good deal of the literature has been devoted to logarithmic or exact asymptotics for first passage probabilities, using different techniques. Martin-Löf [10] and Collamore [7] derived large deviation results for first passage probabilities of a general class of processes. Employing two-dimensional renewal theory and asymptotic properties of ladder processes, respectively, Höglund [9] and von Bahr [3] obtained exact asymptotics for ruin probabilities of the classical risk process (see also Asmussen [2]). Bertoin and Doney [5] generalised the classical Cramér-Lundberg approximation (of the perpetual ruin probability of a classical risk process) to general Lévy processes.

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In this paper we obtain the exact asymptotics of the finite time ruin probability $P(\tau(x) \leq t)$, where $\tau(x) = \inf\{t \geq 0 : X(t) > x\}$, for a general Lévy process $X(t)$ ($X(0) = 0$), if x and t jointly tend to infinity in fixed proportion, generalising Arfwedson [1] and Höglund [9] who treated the case of a classical risk process. The proof is based on an embedding of the ladder process of X and a two-dimensional renewal theorem of Höglund [9].

The remainder of the paper is organized as follows. In Section 2 the main result is presented, and its proof is given in Section 3.

2 Main result

Let X be a Lévy process with non-monotone paths that satisfies

$$E[\mathrm{e}^{\alpha_0 X(1)}] < \infty \quad \text{for some } \alpha_0 > 0, \tag{2.1}$$

and denote by $\tau(x) = \inf\{t \geq 0 : X(t) > x\}$ the first crossing time of x. We exclude the case that X is a compound Poisson process with non-positive infinitesimal drift, as this corresponds to the random walk case which has already been treated in the literature.

The law of X is determined by its Laplace exponent $\psi(\theta) = \log E[\mathrm{e}^{\theta X(1)}]$ that is well defined on the maximal domain $\Theta = \{\theta \in \mathbb{R} : \psi(\theta) < \infty\}$. Restricted to the interior Θ^o, the map $\theta \mapsto \psi(\theta)$ is convex and differentiable, with derivative $\psi'(\theta)$.[1] Moreover, $\psi'(0+) = E[X(1)]$ if $E[|X(1)|] < \infty$. By the strict convexity of ψ, it follows that ψ' is strictly increasing on $(0, \infty)$ and we denote by $\Gamma : \psi'(0, \infty) \to (0, \infty)$ its right-inverse function.

Associated to the measure P is the exponential family of measures $\{P^{(c)} : c \in \Theta\}$ defined by their Radon-Nikodym derivatives

$$\left.\frac{\mathrm{d}P^{(c)}}{\mathrm{d}P}\right|_{\mathcal{F}_t} = \exp\left(cX(t) - \psi(c)t\right). \tag{2.2}$$

It is well known that under this change of measure X is still a Lévy processes and its new Laplace exponent satisfies

$$\psi^{(c)}(\alpha) = \psi(\alpha + c) - \psi(c). \tag{2.3}$$

Related to X and its running supremum are the local time L of X at its supremum, its right-continuous inverse L^{-1} and the upcrossing ladder

[1] For $\theta \in \Theta \backslash \Theta^o$, $\psi'(\theta)$ is understood to be $\lim_{\eta \to \theta, \eta \in \Theta^o} \psi'(\eta)$.

process H respectively. The Laplace exponent κ of the bivariate (possibly killed) subordinator (L^{-1}, H),

$$e^{-\kappa(\alpha,\beta)t} = E[e^{-\alpha L_t^{-1} - \beta H_t}\mathbf{1}_{(L_t^{-1}<\infty)}], \tag{2.4}$$

is related to ψ via the Wiener-Hopf factorisation identity

$$u - \psi(\theta) = k\kappa(u, -\theta)\widehat{\kappa}(u, \theta), \quad u \geq 0, \theta \in \Theta^o, \tag{2.5}$$

for some constant $k > 0$ where $\widehat{\kappa}$ is the Laplace exponent of the dual ladder process. Refer to Bertoin [4, Ch. VI] for further background on the fluctuation theory of Lévy processes.

Bertoin and Doney [5] showed that, if the Cramér condition holds, that is $\gamma > 0$, where

$$\gamma := \sup\{\theta \in \Theta : \psi(\theta) = 0\}, \tag{2.6}$$

the Cramér-Lundberg approximation remains valid for a general Lévy process:

$$\lim_{x\to\infty} e^{\gamma x} P[\tau(x) < \infty] = C_\gamma, \tag{2.7}$$

where $C_\gamma \geq 0$ is positive if and only if $E[e^{\gamma X(1)}|X(1)|] < \infty$ and is then given by $C_\gamma = \beta_\gamma/[\gamma m_\gamma]$, where

$$\beta_\gamma = -\log P[H_1 < \infty], \quad m_\gamma = E[e^{\gamma H_1} H_1 \mathbf{1}_{(H_1<\infty)}].$$

Further, Doob's optional stopping theorem implies the following bound:

$$e^{\gamma x} P(\tau(x) < \infty) = E^{(\gamma)}[e^{-\gamma(X(\tau(x))-x)}\mathbf{1}_{(\tau(x)<\infty)}] \leq 1. \tag{2.8}$$

The result below concerns the asymptotics of the finite time ruin probability $P(\tau(x) \leq t)$ when x, t jointly tend to infinity in fixed proportion. For a given proportion v the rate of decay is either equal to $\gamma v t$ or to $\psi^*(v)t$, where ψ^* is the convex conjugate of ψ:

$$\psi^*(u) = \sup_{\alpha\in\mathbb{R}}(\alpha u - \psi(\alpha)).$$

We restrict ourselves to Lévy processes satisfying the following condition

$$\sigma > 0 \text{ or the Lévy measure is non-lattice,} \tag{H}$$

where σ denotes the Gaussian coefficient of X. Recall that a measure is called non-lattice if its support is not contained in a set of the form $\{a + bh, h \in \mathbb{Z}\}$, for some $a, b > 0$. Note that (H) is satisfied by any Lévy process whose Lévy measure has infinite mass.

We write $f \sim g$ if $\lim_{x,t\to\infty, x=vt+o(t^{1/2})} f(x,t)/g(x,t) = 1$.

Theorem 1 *Assume that* (H) *holds. Suppose that* $0 < \psi'(\gamma) < \infty$ *and that there exists a* $\Gamma(v) \in \Theta^\circ$ *such that* $\psi'(\Gamma(v)) = v$. *If* x *and* t *tend to infinity such that* $x = vt + \mathrm{o}(t^{1/2})$ *then*

$$P(\tau(x) \leq t) \sim \begin{cases} C_\gamma \mathrm{e}^{-\gamma x}, & \text{if } 0 < v < \psi'(\gamma), \\ D_v t^{-1/2} \mathrm{e}^{-\psi^*(v)t}, & \text{if } v > \psi'(\gamma), \end{cases}$$

with $C_0 = 1$ *and* D_v *given by*

$$D_v = \frac{-v \log E[\mathrm{e}^{-\eta_v L_1^{-1}} \mathbf{1}_{(L_1^{-1} < \infty)}]}{\eta_v E[\mathrm{e}^{\Gamma(v) H_1 - \eta_v L_1^{-1}} H_1 \mathbf{1}_{(L_1^{-1} < \infty)}]} \times \frac{1}{\Gamma(v)\sqrt{2\pi\psi''(\Gamma(v))}},$$

where $\eta_v = \psi(\Gamma(v))$.

Remark 1 (a) For a spectrally negative Lévy process the joint exponent of the ladder process is given by $\kappa(\alpha, \beta) = \beta + \Phi(\alpha)$ $(\alpha, \beta \geq 0)$, where $\Phi(\alpha)$ is the largest root of $\psi(\theta) = \alpha$, and thus

$$D_v = \overline{D}_v := \frac{v}{\psi(\Gamma(v))\sqrt{2\pi\psi''(\Gamma(v))}}, \quad C_\gamma \equiv 1. \tag{2.9}$$

Indeed,

$$\begin{aligned} D_v &= \overline{D}_v \times \frac{\kappa(\eta_v, 0)}{\Gamma(v) \frac{\partial}{\partial \beta} \kappa(\eta_v, \beta)_{|\beta = -\Gamma(v)} \exp\{-\kappa(\eta_v, -\Gamma(v))\}} \\ &= \overline{D}_v \times \frac{1}{\exp\{-\Phi(\eta_v) + \Gamma(v)\}} = \overline{D}_v \end{aligned}$$

since $\Phi(\eta_v) = \Gamma(v)$.

(b) If X is spectrally positive, $\kappa(\alpha, \beta) = [\alpha - \psi(-\beta)]/[\hat{\Phi}(\alpha) - \beta]$ (see e.g. [4, Thm VII.4]), where $\hat{\Phi}(\alpha)$ is the largest root of $\psi(-\theta) = \alpha$ and we find that

$$D_v = \frac{\Gamma(v) + \tilde{\Gamma}(v)}{\Gamma(v)\tilde{\Gamma}(v)} \frac{1}{\sqrt{2\pi\psi''(\Gamma(v))}}, \quad C_\gamma = \frac{\psi'(0)}{\psi'(\gamma)},$$

where $\tilde{\Gamma}(v) = \sup\{\theta : \psi(-\theta) = \psi(\Gamma(v))\}$, recovering formulas that can be found in Arfwedson [1] and Feller [8] respectively, for the case of a classical risk process.

Remark 2 Heuristically, in the case $v > \psi'(\gamma)$, the asymptotics in Thm. 1 can be regarded as a consequence of the central limit theorem, that is, under the tilted measure $P^{\Gamma(v)}$, asymptotically

$$\frac{\tau(x) - x/v}{\omega\sqrt{x}}$$

follows a standard normal distribution, where by (2.3) and choice of $\Gamma(v)$,

$$\omega^2 = \frac{\mathrm{Var}^{(\Gamma(v))}[X_1]}{\left(E^{(\Gamma(v))}[X_1]\right)^3} = \frac{\psi^{(\Gamma(v))\prime\prime}(0)}{\left(\psi^{(\Gamma(v))\prime}(0)\right)^3} = \frac{\psi''(\Gamma(v))}{v^3}.$$

This explains why the asymptotics remain valid if x deviates $\mathrm{o}(x^{1/2}) = \mathrm{o}(t^{1/2})$ from the line vt.

In the boundary case $v = \psi'(\gamma)$, in which case $E^{(\Gamma(v))}[\tau(x)] = t$, the exact asymptotics of $P(\tau(x) \leq t)$ may depend on the way in which x/t tends to v. Note that this case is excluded from Theorem 1.

Remark 3 In the case $0 < v < \psi'(\gamma)$, the asymptotics in Theorem 1 are a consequence of the law of large numbers. To see why this is the case, note that $\mathrm{e}^{\gamma x} P(\tau(x) \leq t) = \mathrm{e}^{\gamma x} P(\tau(x) < \infty) - \mathrm{e}^{\gamma x} P(t < \tau(x) < \infty)$, where the first term tends to C_γ in view of (2.7), while for the second term the Markov property and (2.8) imply that

$$\begin{aligned}
&\mathrm{e}^{\gamma x} P(t < \tau(x) < \infty) \\
&\quad = \int_{-\infty}^{x} P(\tau(x) > t, X(t) \in dy) \mathrm{e}^{\gamma y} \mathrm{e}^{\gamma(x-y)} P(\tau(x-y) < \infty) \\
&\quad \leq \int_{-\infty}^{x} P(X(t) \in dy) \mathrm{e}^{\gamma y} = P^{(\gamma)}(X(t) \leq x),
\end{aligned}$$

which tends to 0 as t tends to infinity in view of the law of large numbers since $E^{(\gamma)}[X(t)] = t\psi'(\gamma) > x$. The proof below deals with the case that $v > \psi'(\gamma)$.

3 Proof of Theorem 1

The idea of the proof is to lift asymptotic results that have been established for random walks by Höglund [9] and Arfwedson [1] to the setting of Lévy processes by considering suitable random walks embedded in the Lévy process (more precisely, in its ladder process). We first briefly recall these results following the Höglund [9] formulation.

3.1 Review of Höglund's random walk asymptotics

Let $(S,R) = \{(S_i,R_i), i = 1,2,\ldots\}$ be a (possibly killed) random walk starting from $(0,0)$ whose components S and R have non-negative increments, and consider the crossing probabilities

$$\begin{aligned} G_{a,b}(x,y) &= P(N(x) < \infty, S_{N(x)} > x+a, R_{N(x)} \leq y+b), \\ K_{a,b}(x,y) &= P(N(x) < \infty, S_{N(x)} > x+a, R_{N(x)} \geq y+b), \end{aligned}$$

where $a \geq 0, b \in \mathbb{R}$ and $N(x) = \min\{n : S_n > x\}$. Let F denote the (possibly defective) distribution function of the increments of the random walk with joint Laplace transform ϕ and set $F_{(u,v)}(\mathrm{d}x,\mathrm{d}y) = \mathrm{e}^{-ux-vy}F(\mathrm{d}x,\mathrm{d}y)/\phi(u,v)$. Let

$$V(\zeta) = E_\zeta[(R_1E_\zeta[S_1] - S_1E_\zeta[R_1])^2]/E_\zeta[S_1]^3$$

for $\zeta = (\xi,\eta)$ where E_ζ denotes the expectation w.r.t. F_ζ.

For our purposes it will suffice to consider random walks that satisfy the following non-lattice assumption (the analogue of the non-lattice assumption in one dimension):

$$\text{The additive group spanned by the support of } F \text{ contains } \mathbb{R}^2_+. \tag{G}$$

Specialised to our setting Prop. 3.2 in Höglund (1990) jointly with the remark given on p. 380 therein read as follows:

Proposition 1 *Assume that (G) holds, and that there exists a $\zeta = (\xi,\eta)$ with $\phi(\zeta) = 1$ such that $v = E_\zeta[S_1]/E_\zeta[R_1]$, where ϕ is finite in a neighbourhood of ζ and $(0,\eta)$. If x,y tend to infinity such that $x = vy + \mathrm{o}(y^{1/2}) > 0$ then it holds that*

$$\begin{aligned} G_{a,b}(x,y) &\sim D(a,b)x^{-1/2}\mathrm{e}^{x\xi+y\eta} && \textit{if } \eta > 0, \\ K_{a,b}(x,y) &\sim D(a,b)x^{-1/2}\mathrm{e}^{x\xi+y\eta} && \textit{if } \eta < 0, \end{aligned}$$

for $a \geq 0, b \in \mathbb{R}$, where $D(a,b) = C(a,b)\cdot(2\pi V(\zeta))^{-1/2}$, with $V(\zeta) > 0$ and

$$C(a,b) = \frac{1}{|\eta|E_\zeta[S_1]}\mathrm{e}^{b\eta}\int_a^\infty P_\zeta(S_1 \geq x)\mathrm{e}^{\xi x}\mathrm{d}x.$$

3.2 Embedded random walk

Denote by $e_1, e_2, \ldots$ a sequence of independent $\exp(q)$ distributed random variables and by $\sigma_n = \sum_{i=1}^n e_i$, with $\sigma_0 = 0$, the corresponding partial sums,

and consider the two-dimensional (killed) random walk $\{(S_i, R_i), i = 1, 2 \dots\}$ starting from $(0,0)$ with step-sizes distributed according to

$$F^{(q)}(\mathrm{d}t, \mathrm{d}x) = P(H_{\sigma_1} \in \mathrm{d}x, L^{-1}_{\sigma_1} \in \mathrm{d}t),$$

and write $G^{(q)}$ for the corresponding crossing probability

$$G^{(q)}(x, y) = G_{0,0}(x, y) = P(N(x) < \infty, R_{N(x)} \leq y).$$

Note that $F^{(q)}$ is a probability measure that is defective precisely if X drifts to $-\infty$, with Laplace transform ϕ given by

$$\phi(u, v) = \iint \mathrm{e}^{-ut-vx} F^{(q)}(\mathrm{d}t, \mathrm{d}x) = \frac{q}{q - \kappa(u, v)}.$$

The key step in the proof is to derive bounds for $P(\tau(x) \leq t)$ in terms of crossing probabilities involving the random walk (S, R):

Lemma 1 *Let $M, q > 0$. For $x, t > 0$ it holds that*

$$G^{(q)}(x, t) \leq P(\tau(x) \leq t) \leq G^{(q)}(x, t + M)/h(0-, M), \tag{3.1}$$

where $h(0-, M) = \lim_{x\uparrow 0} h(x, M)$, with $h(x, t) := P(H_{\sigma_1} > x, L^{-1}_{\sigma_1} \leq t)$.

Proof: Let $T(x) = \inf\{t \geq 0 : H_t > x\}$ and note that $\tau(x) = L^{-1}_{T(x)}$. By applying the Markov property it follows that

$$\begin{aligned}
P(\tau(x) \leq t) &= P(T(x) < \infty, L^{-1}_{T(x)} \leq t) \\
&= \sum_{n=1}^{\infty} P(\sigma_{n-1} \leq T(x) < \sigma_n, L^{-1}_{T(x)} \leq t) \qquad (3.2)\\
&= \sum_{n=1}^{\infty} P(H_{\sigma_{n-1}} \leq x, H_{\sigma_n} > x, L^{-1}_{T(x)} \leq t) \\
&= \sum_{n=1}^{\infty} \iint P(H_{\sigma_{n-1}} \in \mathrm{d}y, L^{-1}_{\sigma_{n-1}} \in \mathrm{d}s) \\
&\qquad \times P(H_{\sigma_1} > x - y, L^{-1}_{T(x-y)} \leq t - s) \qquad (3.3)\\
&= \sum_{n=0}^{\infty} F^{(q)\star n} \star f(x, t) = (U \star f)(x, t), \qquad (3.4)
\end{aligned}$$

where $U = \sum_{n=0}^{\infty} F^{(q)\star n}$, $f(x, t) = P(H_{\sigma_1} > x, L^{-1}_{T(x)} \leq t)$ and $\star$ denotes convolution. Following a similar reasoning it can be checked that

$$G^{(q)}(x, t) = U \star h(x, t). \tag{3.5}$$

In view of (3.4) and (3.5), the lower bound in (3.1) follows since

$$f(x,t) \geq h(x,t),$$

taking note of the fact that $H_{\sigma_1} > x$ precisely if $T(x) < \sigma_1$, while the upper bound in (3.1) follows by observing that for fixed $M > 0$,

$$\begin{aligned} h(x,t+M) &\geq P(H_{\sigma_1} > x, L^{-1}_{T(x)} \leq t, L^{-1}_{\sigma_1} - L^{-1}_{T(x)} \leq M) \\ &= P(H_{\sigma_1} > x, L^{-1}_{T(x)} \leq t) P(L^{-1}_{\sigma_1} \leq M) \\ &= f(x,t)h(0-,M), \end{aligned}$$

where we used the strong Markov property of L^{-1} and the lack of memory property of σ_1. □

Applying Höglund's asymptotics in Proposition 1 yields the following result:

Lemma 2 *Let the assumptions of Proposition 1 hold true. If $x,t \to \infty$ such that for $v > \psi'(\gamma)$ we have $x = vt + \mathrm{o}(t^{1/2})$ then*

$$G^{(q)}(x,t+M) \sim D_{q,M} t^{-1/2} \mathrm{e}^{-\psi^*(v)t}, \quad M \geq 0,$$

where $D_{q,M} = \frac{v}{\sqrt{2\pi\psi''(\Gamma(v))}} C_{q,M}$ with

$$C_{q,M} = \mathrm{e}^{\psi(\Gamma(v))M} \frac{\kappa(\psi(\Gamma(v)),0)}{c_v \psi(\Gamma(v))\Gamma(v)} \frac{q}{q + \kappa(\psi(\Gamma(v)),0)},$$

where $c_v = E[\mathrm{e}^{\Gamma(v)H_1 - \psi(\Gamma(v))L_1^{-1}} H_1 \mathbf{1}_{(L_1^{-1} < \infty)}]$.

Lemma 2 is a consequence of the following auxiliary identities:

Lemma 3 *Let $u > \gamma$, $u \in \Theta^o$.*

$$\begin{aligned} \phi(z,-u) &= 1 \quad \text{iff } \kappa(z,-u) = 0 \text{ iff } \psi(u) = z && (3.6) \\ \psi'(u) &= E^{(u)}[X(1)] = E^{(u)}[H_{\sigma_1}] \cdot (E^{(u)}[L^{-1}_{\sigma_1}])^{-1} && (3.7) \\ \psi''(u) &= E^{(u)}[(H_{\sigma_1} - \psi'(u)L^{-1}_{\sigma_1})^2] \cdot (E^{(u)}[L^{-1}_{\sigma_1}])^{-1} \\ &= \psi'(u)E^{(u)}[(H_{\sigma_1} - \psi'(u)L^{-1}_{\sigma_1})^2] \cdot (E^{(u)}[H_{\sigma_1}])^{-1} && (3.8) \\ \psi^*(v) &= v\Gamma(v) - \psi(\Gamma(v)) \quad \text{for } v > 0 \text{ with } \Gamma(v) \in \Theta^o. && (3.9) \end{aligned}$$

Proof: Eq (3.6): Note that for $u,z > 0$ it holds that $\widehat{\kappa}(z,u) > 0$. In view of the identity (2.5) the statement follows.

Eq (3.7): Note that if $u > \gamma$ then by the fact that $\psi(0) = \psi(\gamma) = 0$ and the strict convexity of ψ it follows that $\psi(u) > 0$. In view of (2.5) it follows then that $\kappa(\psi(u), -u) = 0$ for $u \in \Theta^o$, $u > \gamma$. Differentiating with respect to u shows that

$$\psi'(u) = \partial_2\kappa(\psi(u), -u)(\partial_1\kappa(\psi(u), -u))^{-1}. \tag{3.10}$$

Also, note that $E^{(u)}[H_{\sigma_1}] = q^{-1}E^{(u)}[H_1]$, $E^{(u)}[L_{\sigma_1}^{-1}] = q^{-1}E^{(u)}[L_1^{-1}]$ and

$$E^{(u)}[H_1] = \partial_2\kappa(\psi(u), -u), \quad E^{(u)}[L_1^{-1}] = \partial_1\kappa(\psi(u), -u).$$

Eq (3.8) follows as a matter of calculus, by differentiation of (3.10) with respect to u. Finally, Eq. (3.9) follows from the definition of ψ^*. □

Proof of Lemma 2 The proof follows by an application of Prop. 1 to $G^{(q)}(x, t + M)$ with

$$(S_1, R_1) = (H_{\sigma_1}, L_{\sigma_1}^{-1}) \qquad \text{and} \qquad \zeta = (-\Gamma(v), \eta_v).$$

Note that, by (3.6) with $u = \Gamma(v)$, $\phi(\zeta) = 1$, and that $\eta_v = \psi(\Gamma(v)) > 0$ if $v > \psi'(\gamma)$. For this choice of the parameters, $E_\zeta[S_1] = E^{(\Gamma(v))}[H_{\sigma_1}] = c_v/q$, and Eqs. (3.9), (3.7),(3.8) imply that $\xi x + \eta t = -\psi^*(v)t$ and

$$V(\zeta) = \psi''(\Gamma(v))/\psi'(\Gamma(v)) = \psi''(\Gamma(v))/v.$$

To complete the proof we are left to verify the form of the constants. The calculation of the $C_{q,M} = C(0,0)e^{\eta M}$ goes as follows:

$$\begin{aligned}
C_{q,M} &= \frac{qe^{\psi(\Gamma(v))M}}{\psi(\Gamma(v))c_v}\left(\int_0^\infty e^{-\Gamma(v)x}E[e^{\Gamma(v)H_{\sigma_1}-\psi(\Gamma(v))L_{\sigma_1}^{-1}}\mathbf{1}_{(x\le H_{\sigma_1}<\infty)}]dx\right)\\
&= \frac{qe^{\psi(\Gamma(v))M}}{\psi(\Gamma(v))\Gamma(v)c_v}\left(1 - E[e^{-\psi(\Gamma(v))L_{\sigma_1}^{-1}}\mathbf{1}_{(L_{\sigma_1}^{-1}<\infty)}]\right)\\
&= \frac{qe^{\psi(\Gamma(v))M}}{\psi(\Gamma(v))\Gamma(v)c_v}\left(1 - \frac{q}{q+\kappa(\psi(\Gamma(v)),0)}\right)\\
&= \frac{qe^{\psi(\Gamma(v))M}}{\psi(\Gamma(v))\Gamma(v)c_v}\frac{\kappa(\psi(\Gamma(v)),0)}{q+\kappa(\psi(\Gamma(v)),0)},
\end{aligned}$$

in view of the definition (2.4) of κ. Combining all results completes the proof. □

As final preparation for the proof of Theorem 1 we show that the non-lattice condition holds:

Lemma 4 *Suppose that (H) holds true. Then $F^{(q)}$ satisfies (G).*

Proof: The assertion is a consequence of the following identity between measures on $(0,\infty)^2$ (which is itself a consequence of the Wiener-Hopf factorisation, see e.g. Bertoin [4, Cor VI.10])

$$P(X_t \in \mathrm{d}x)\mathrm{d}t = t \int_0^\infty P(L_u^{-1} \in \mathrm{d}t, H_u \in \mathrm{d}x)u^{-1}\mathrm{d}u. \tag{3.11}$$

Fix $(y,v) \in (0,\infty)^2$ in the support of $\mu_X(\mathrm{d}t,\mathrm{d}x) = P(X_t \in \mathrm{d}x)\mathrm{d}t$ and let B be an arbitrary open ball around (y,v). Then $\mu_X(B) > 0$; in view of the identity (3.11) it follows that there exists a set A with positive Lebesgue measure such that $P((L_u^{-1}, H_u) \in B) > 0$ for all $u \in A$ and thus $P((L_{\sigma_1}^{-1}, H_{\sigma_1}) \in B) > 0$. Since B was arbitrary we conclude that (y,v) lies in the support of $F^{(q)}$. To complete the proof we next verify that if a Lévy process X satisfies (H) then μ_X satisfies (G). To this end, let X satisfy (H). Suppose first that its Lévy measure ν has infinite mass or $\sigma > 0$. Then $P(X_t = x) = 0$ for any $t > 0$ and $x \in \mathbb{R}$, according to Sato [14, Thm. 27.4]. Thus, the support of $P(X_t \in \mathrm{d}x)$ is uncountable for any $t > 0$, so that μ_X satisfies (G). If ν has finite mass then it is straightforward to verify that $P(X_t \in \mathrm{d}x)$ is non-lattice for any $t > 0$ if ν is, and that then μ_X satisfies (G). □

Proof of Theorem 1: Suppose that $v > \psi'(\gamma)$ (the case $v < \psi'(\gamma)$ was shown in Remark 3). Writing $l(t,x) = t^{1/2}\mathrm{e}^{\psi^*(v)t}P(\tau(x) \le t)$, Lemmas 1, 2 and 3 imply that

$$\begin{aligned} s &= \limsup_{x,t\to\infty, x=tv+\mathrm{o}(t^{1/2})} l(t,x) \le D_{q,M}/h(0-,M), \\ i &= \liminf_{x,t\to\infty, x=tv+\mathrm{o}(t^{1/2})} l(t,x) \ge D_{q,0}. \end{aligned}$$

By definition of h and $D_{q,M}$ it directly follows that, as $q \to \infty$,

$$D_{q,0} \to D_v,\ D_{q,M} \to D_v\mathrm{e}^{\psi(\Gamma(v))M} \text{ and } h(0-,M) = P(L_{\sigma_1}^{-1} \le M) \to 1.$$

Letting $M \downarrow 0$ yields that $s = i = D_v$, and the proof is complete. □

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